WEYL'S LAW ON RIEMANNIAN MANIFOLDS

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ABSTRACT. We motivate Weyl's law by demonstrating the relevance of the distribution of eigenvalues of the Laplacian to the ultraviolet catastrophe of physics. We then introduce Riemannian geometry, define the Laplacian on a general Riemannian manifold, and find geometric analogs of various concepts in $\mathbb{R}^n$, along the way using $\mathbb{S}^2$ as a clarifying example. Finally, we use analogy with $\mathbb{R}^n$ and the results we have built up to prove Weyl's law, making sure at each step to demonstrate the physical significance of any major ideas.

CONTENTS

1. THE ROLE OF WEYL'S LAW IN THE ULTRAVIOLET CATASTROPHE

In the year 1900 Lord Rayleigh used the equipartition theorem of thermodynamics to deduce the famous Rayleigh-Jeans law of radiation. Rayleigh began with an idealized physical concept of the blackbody; a body which is a perfect absorber of electromagnetic radiation and which radiates all the energy it absorbs independent of spatial direction. We sketch his proof to find the amount of radiation energy emitted by the blackbody at a given frequency.

Take a blackbody in the shape of a cube for definiteness, and assume it is made of conducting material and is at a temperature T. We will denote the cube by $D = [0, L]^3 \subseteq \mathbb{R}^3$. Inside this cube we have electromagnetic radiation that obeys Maxwell's equations bouncing around. The equipartition theorem of thermodynamics says that every wave which has constant spatial structure with oscillating amplitude, i.e. a normal mode, possesses energy $k_B T$ where k_B is the Boltzmann constant [6]. If we wish to find the energy contained as radiation in the blackbody box between the frequencies ν and $\nu + \mathrm{d}\nu$, we therefore have to count the number of normal modes with frequencies between ν and $\nu + \mathrm{d}\nu$ and multiply by $k_B T$.

Let $\mathbf{E}_k(\mathbf{r}, t)$ describe the electric field of the kth normal mode as a function of space and time. We may thus write $\mathbf{E}_k(\mathbf{r}, t) = u_k(\mathbf{r}) g_k(t) \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is an arbitrary spatial direction, as this is the mathematical definition of a normal mode. Further, since $\mathbf{E}_k$ satisfies Maxwell's

Date: August 29, 2016.

equations we may use a standard separation of variables argument to simplify the problem. This allows us to deduce that

$$\Delta_{\mathbb{R}^3} u_k = \lambda_k u_k \text{ for some } \lambda_k \in \mathbb{R}_{\geq 0},\ \Delta_{\mathbb{R}^3} = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \text{ (Laplacian on } \mathbb{R}^3\text{)},$$

and that $\lambda_k = 4\pi^2\nu_k^2/c^2$, where ν_k is the frequency of the kth normal mode and c is the speed of light. Additionally since the blackbody is conducting, the electric field must be zero at the boundary, so $u_k \mid_{\partial D}= 0$.

Counting the frequencies of normal modes between ν and $\nu + \mathrm{d}\nu$ is therefore reduced to counting the eigenvalues of the Laplacian on our blackbody. A second separation of variables reveals that the distribution of eigenvalues on the cube D grows asymptotically as

$$N(\lambda) = |\{k \in \mathbb{N} \mid \lambda_k \leq \lambda\}| \sim \frac{1}{6\pi^2} L^3 \lambda^{3/2} = \frac{1}{6\pi^2}\mathrm{Vol}(D)\lambda^{3/2}.$$

Thus if we wish to find the number of normal modes with frequencies between ν and $\nu + \mathrm{d}\nu$ we may use the fact mentioned above that $\lambda_k = 4\pi^2\nu_k^2/c^2$ and the form of $N(\lambda)$ to see

$$|\{k \in \mathbb{N} \mid \nu \leq \nu_k \leq \nu + \mathrm{d}\nu\}| = N\left(\frac{4\pi^2}{c^2}(\nu + \mathrm{d}\nu)^2\right) - N\left(\frac{4\pi^2}{c^2}\nu^2\right) \sim \frac{4\pi \mathrm{Vol}(D)}{c^3}\nu^2 \mathrm{d}\nu.$$

We have assumed ν is large enough to safely apply asymptotics, an acceptable assumption experimentally ([1], Section 1.2.3). We can then conclude by the equipartition theorem that the amount of energy, contained as radiation, in our blackbody per unit volume between the frequency ν and $\nu + \mathrm{d}\nu$ is

$$\rho(\nu) = 8\pi k_B T \frac{\nu^2}{c^3}\mathrm{d}\nu,$$

where the factor of two comes from taking both polarizations of light into account.

This is known as the classical Rayleigh-Jeans law. Unfortunately it is immediately and spectacularly wrong in the realm of large ν. Note that the energy contained as radiation in the blackbody should be equal to the integral of $\rho(\nu)$ from 0 to ∞, since classically all frequencies should be represented inside the cavity. But the ν^2 dependence of $\rho(\nu)$ ensures that this integral will be infinite, therefore violating the conservation of energy. This problem was dubbed the ultraviolet catastrophe and was solved by Planck's introduction of quantum mechanics [6]. However, before Planck's solution it was an open question as to where Lord Rayleigh went wrong in this derivation of his namesake law. Many physicists were loath to admit that the problem was with the equipartition theorem, so some suspicion fell on the validity of $N(\lambda)$'s asymptotic form for a general region. Indeed, interest in the general asymptotic form of $N(\lambda)$ led Lorentz to give a talk in 1910 laying out the problem [1].

In 1911 Weyl solved the problem, where he showed that if $D \subseteq \mathbb{R}^n$ was *any* compact region and $N(\lambda)$ was the counting function for the eigenvalues of the problem $\Delta u_k = \lambda_k u_k$ with $u_k|_{\partial D} = 0$, then

$$N(\lambda) \sim \frac{|B_n|}{(2\pi)^n}\mathrm{Vol}(D)\lambda^{n/2},$$

where $|B_n|$ is the volume of the n-ball $B_n(0) \subseteq \mathbb{R}^n$ [1]. This result is known as Weyl's law. In fact Weyl's law can be extended to a general compact Riemannian manifold, with the eigenvalues of the Laplacian giving information about the boundary measure and total curvature of the Riemannian manifold as well as its volume. Not only is Weyl's law a fascinating piece of mathematics, but it also indicates very strongly that the error in Rayleigh's argument was not assuming the asymptotic expression $N(\lambda)$ held for a general region, but was rather a more fundamental breakdown of the equipartition theorem. Indeed just such a breakdown does occur at high frequencies, though it took the introduction of quantum mechanics to understand why.

In this paper we will seek to understand Weyl's law on a compact Riemannian manifold, allowing us to get insight into a result caught up in the ultraviolet catastrophe. We will attempt to develop everything necessary to prove Weyl's law in a way accessible to the reader who has only had instruction in smooth manifolds at the level of Lee [**?**]. Before we can dive into the proof we first must develop an understanding of what the Laplacian means on a general Riemannian manifold, and indeed what exactly a Riemannian manifold is. We refer to Lee [4] for the basic results and notation of smooth manifold theory, for example notation on tensor or vector fields.

Our first task will be to establish some basic results of Riemannian geometry, including definitions and basic examples. We will then explore some more advanced topics, including the definition of the Laplacian on a general Riemannian manifold and some basic results about geodesics and normal coordinates on our manifold. After this introduction we will be ready to discuss the proof of Weyl's law, covering all necessary lemmas along the way. Some of the proofs in the section on Weyl's law are long out of necessity, but the more casual reader can safely skip some details without sacrificing too much clarity.

2. Riemannian Manifolds

2.1. **Basic Results.**

Definition 2.1. A Riemannian metric on a smooth manifold M is a $(0, 2)$ tensor field $g \in \mathcal{T}^2(M)$ such that $g(X, Y) = g(Y, X)$ and $g(X, X) > 0$ for any $X, Y \in \mathfrak{X}(M)$ with $X \neq 0$. The collection (M, g) is called a Riemannian manifold, though if g is obvious from context we will often refer simply to M as a Riemannian manifold.

Throughout this paper unless otherwise specified M will be a Riemannian manifold, with metric g, and with M oriented, compact, and connected. Compactness and connectedness are clear concepts from point set topology, and a discussion of what it means to be an oriented manifold, as well as a discussion of oriented local frames, can be found in Chapter 15 of [4].

It is often helpful when doing computations on a manifold to work in coordinates. If $(U, (x^i))$ is a local coordinate chart for the Riemannian manifold (M, g) then the metric can be expressed in coordinates as $g = g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j$, where we employ the Einstein summation convention. If we additionaly let the symmetric product of $\omega, \nu \in \Omega^1(M)$ be denoted by $\omega\nu$ then by the symmetry of g we can write $g = g_{ij} \mathrm{d}x^i \mathrm{d}x^j$. In this case we have the useful relationship

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right),$$

in coordinates.

Example 2.2. There are two very canonical examples.

(a) Take $a \in \mathbb{R}^n$ and $T_a\mathbb{R}^n \simeq \mathbb{R}^n$, with the canonical isomorphism, to be its tangent space. Then we take $g_a(v, w) = v \cdot w$ for $v, w \in T_a\mathbb{R}^n$, where $\cdot$ is the standard Euclidean dot product. This clearly is symmetric and positive definite on each tangent space, and can be extended to a Riemannian metric g on $\mathbb{R}^n$. We call this the Euclidean metric. Note that in canonical coordinates we have that $g(\partial_{x^i}, \partial_{x^j}) = \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, where δ_{ij} is the Kronecker delta. Then $g = \sum_{i=1}^n (\mathrm{d}x^i)^2$.

Alternative local coordinates for $\mathbb{R}^3$ at points away from zero are the spherical coordinates, defined by

$$(x^1, x^2, x^3) = (r\cos\phi\sin\theta,\ r\sin\phi\sin\theta,\ r\cos\theta).$$

In this coordinate system, since g transforms like any other 2-tensor field, after some calculation we have that

$$g = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 = \mathrm{d}r^2 + r^2\ \mathrm{d}\theta^2 + r^2\sin^2\theta\ \mathrm{d}\phi^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2),$$

which is the Euclidean metric in spherical coordinates.

(b) Suppose that S is a submanifold of M with inclusion map $\iota\colon S \hookrightarrow M$. If (M, g^M) is a Riemannian manfiold then we can give S a Riemannian metric by taking $g^S = \iota^* g^M$. Since the pullback of a $(0,2)$-tensor field gives a $(0,2)$-tensor field, and since $\mathrm{d}\iota_p$ is an injective map for all $p \in M$, we know that g^S is a symmetric and positive definite $(0,2)$-tensor field. This makes (S, g^S) into a Riemannian manifold.

The canonical example of this is $\mathbb{S}^2 \subseteq \mathbb{R}^3$ where $\mathbb{R}^3$ is given the standard Euclidean metric. Since $\mathbb{S}^2$ is the submanifold of $\mathbb{R}^3$ with r restricted to 1 in spherical coordinates, the pullback of g as in (a) will give us that

$$g = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$$

is the pullback of the Euclidean metric to $\mathbb{S}^2$, where

$$T\colon (0,\pi) \times (0, 2\pi) \to \mathbb{R}^3\colon (\theta,\phi) \mapsto (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

is a smooth local parametrization of $\mathbb{S}^2 \subseteq \mathbb{R}^3$.

We will frequently look back to $\mathbb{S}^2$ in order to give our remarks footing in a manifold that is familiar, but not trivial, as is often the case with $\mathbb{R}^n$.

The additional structure given by a Riemannian metric lets us extend many helpful concepts from $\mathbb{R}^n$ to M. First we write the inner product given by g as $\langle\cdot,\cdot\rangle_g$. We also define $|v|_g = \sqrt{\langle v, v\rangle_g}$ to be the length of the vector $v \in T_pM$, where we note that $|\cdot|_g$ defines a norm on T_pM for all $p \in M$. We can use this norm to extend the concept of the length of a curve in $\mathbb{R}^n$.

Definition 2.3. Let $\gamma\colon [a,b] \to M$ be a piecewise smooth curve in M and define

$$L_g(\gamma) = \int_a^b |\gamma'(t)|_g\ \mathrm{d}t$$

to be the length of γ.

Further let $x, y \in M$ and define

$$d_g(x,y) = \inf\{L_g(\gamma) \mid \gamma \text{ piecewise smooth } \gamma(a) = x,\ \gamma(b) = y\}.$$

If γ is a curve with endpoints x and y such that $d_g(x,y) = L(\gamma)$ then we call γ a minimizing curve. On a general Riemannian manifold such a curve might not exist.

Since we have taken M to be connected, it is path connected, and so d_g will always be defined. In fact d_g gives a metric to M.

Theorem 2.4. *Let M be a connected Riemannian manifold, then with d_g the manifold M becomes a metric space whose topology is identical to the original manifold topology.*

Proof. This is Theorem 13.29 in [4]. □

It is worth noting that since every compact metric space is complete, then with our choice of M as compact and connected it will be complete with the metric d_g.

In addition to defining a metric on our manifold we can use the inner product $\langle \cdot, \cdot \rangle_g$ to extend our concept of orthonormality to frames.

Definition 2.5. Let $(E_1, \dots, E_n)$ be a local frame for M on some open set $U \subseteq M$. We call this an orthonormal frame if $(E_1 \mid_p, \cdots, E_n \mid_p)$ is an orthonormal basis for T_pM for every $p \in U$.

Remark 2.6. Note that a smooth local orthonormal frame always exists for every Riemannian manifold M. One simply starts with the smooth local frame given by coordinates and performs the Gram-Schimdt process using the inner product $\langle \cdot, \cdot \rangle_g$. Since this is a smooth operation on $\mathfrak{X}(M)$ this does indeed produce a smooth local orthonormal frame.

Additionally if our manifold M has a smooth local orthonormal frame $(E_1, \dots, E_n)$ then if necessary we can replace E_1 by $-E_1$ to get a smooth oriented local orthonormal frame as defined in Chapter 15 of [4].

It turns out the existence of a smooth oriented local orthonormal frame allows us to define an important unique element in $\Omega^n(M)$, where $n = \dim M$.

Proposition 2.7. *If (M, g) is an oriented Riemannian n-manifold then there exists a unique $\omega_g \in \Omega^n(M)$ such that for every local oriented orthonormal frame $(E_1, \dots, E_n)$ we have*

$$\omega_g(E_1, \dots, E_n) = 1.$$

We call this n-form the Riemannian volume form.

Proof. This is Proposition 15.29 in [4]. □

The existence of such a unique form allows us to define integration of a function on M, something we couldn't do on a general manifold.

Definition 2.8. Take our manifold with its volume form ω_g. Let f be a compactly supported continuous function on M, then $f\omega_g$ is a compactly supported n-form and we say that the integral of f over M is $\int_M f\omega_g$. We frequently write this as $\int_M f \mathrm{d}V_g$.

We also define $\mathrm{Vol}(M) = \int_M \mathrm{d}V_g$, allowed because we have taken M to be compact.

From this definition of integration, we can quickly establish an important class of function spaces on M. Tracing the proof that the norm on $L^p(\mathbb{R})$ is in fact a norm for $1 \leq p < \infty$, we can see immediately that

$$\|f\|_{L^p(M)} = \left(\int_M |f|^p \, \mathrm{d}V_g \right)^{1/p}$$

defines a norm on $C^0(M)$ with M compact. We leave this as an exercise for the reader.

Definition 2.9. The space $L^p(M)$ is defined to be the completion of $C^0(M)$ with respect to the norm $\| \cdot \|_{L^p(M)}$.

Note also that $L^2(M)$ can be given the standard inner product of L^2 and turned into a Hilbert space using integration, as defined here.

Though it is clear that the existence of a unique n-form ω_g helps to develop integration of functions on M, it is not really clear why we call ω_g the volume form on M. In order to make this clear, we will give its expression in local coordinates $(U, (x^i))$,

$$\omega_g = \sqrt{\det g}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n.$$

This can be found as Proposition 15.31 in [4], and is straightforward.

Since $g_{ij} = \delta_{ij}$ for $\mathbb{R}^n$ we see that in $\mathbb{R}^n$ we have $\omega_g = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, which corresponds to the usual Euclidean volume form. In the case of $\mathbb{S}^2$ we have that $\det g = 1 \cdot \sin^2\theta$, so that $\omega_g = \sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\phi$. This corresponds to the volume form over which the physicist is familiar with integrating when integrating functions on $\mathbb{S}^2$. Thus at least on these simple manifolds we see that our volume form corresponds to a volume form we are familiar with.

2.2. **Building the Laplacian and its Basic Properties.** We continue to use the Riemannian metric to further extend standard tools such as the gradient and divergence to (M, g). Some observation reveals that if $\hat{g}(v)\colon T_pM \to \mathbb{R}$ is a map defined as $\hat{g}(v)(w) = \langle v, w\rangle_g$ then we will have that $\hat{g}(v) \in T_p^*M$ for every $p \in M$. In particular we see that $\hat{g}$ maps TM to T^*M according to this relation. The linearity of g over $C^\infty(M)$ makes $\hat{g}$ a bundle homomorphism. Further, since g is positive-definite $\hat{g}$ is a bundle isomorphism, so in particular there exists a bundle isomorphism $\hat{g}^{-1}\colon T^*M \to TM$. The existence of this bundle isomorphism allows us to define a notion of a gradient, since we may extend to smooth sections, $\hat{g}^{-1}\colon \Omega^1(M) \to \mathfrak{X}(M)$.

Definition 2.10 (Gradient)**.** Let $f \in C^\infty(M)$, then we define

$$\text{grad } f = \hat{g}^{-1}(\mathrm{d}f),$$

and call grad f the gradient of f.

Note that if $X \in \mathfrak{X}(M)$ then $\langle \text{grad } f, X\rangle_g = \hat{g}\left(\hat{g}^{-1}(\mathrm{d}f)\right)(X) = Xf$. From this fact we have that in local coordinates $(U, (x^i))$

$$\text{grad} f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j},$$

where g^{ij} is the inverse of the matrix g_{ij}. We see that in the case of $\mathbb{R}^n$ this agrees with the familiar gradient, given the identification of each tangent space with $\mathbb{R}^n$. A perhaps less intuitive identification than the gradient is the divergence. Lee [4] uses $\iota\colon \Omega^k(M) \to \Omega^{k-1}(M)$ to denote interior multiplication, so we will as well. Thus if ω is a k-form and X is a vector field, then $\iota_X\omega$ is the $(k-1)$-form $\omega(X, \cdot, \dots, \cdot)$.

Definition 2.11 (Divergence)**.** We know that $d \circ \iota_X\colon \Omega^k(M) \to \Omega^k(M)$, so since ω_g is top-degree there exists some $\text{div}X \in C^\infty(M)$ such that

$$(\mathrm{d} \circ \iota_X)\omega_g = \text{div}X\ \omega_g.$$

We call $\text{div}X$ the divergence of the vector field, and note that $\text{div}\colon \mathfrak{X}(M) \to C^\infty(M)$.

It is straightforward to verify directly from the definition that in local coordinates, $(U, (x^i))$, the divergence of $X = X^i \frac{\partial}{\partial x^i}$ is given by

$$\text{div}X = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(X^i \sqrt{\det g} \right).$$

Note that in $\mathbb{R}^n$ we have $\sqrt{\det g} = 1$, so that this agrees with our standard notion of the divergence on $\mathbb{R}^n$.

From the definition of div and Stoke's theorem we can derive an analog of Gauss's theorem. First we recall what it means for a manifold to have a boundary, and then define a notion of a normal vector. Recall that a manifold with boundary is simply a manifold having coordinate charts whose homeomorphisms are allowed to map into relatively open subsets of the half-plane $\mathbb{H}^n$, as opposed to $\mathbb{R}^n$. The boundary of the manifold M consists of all points which map to the boundary of $\mathbb{H}^n$ under these charts, and smoothness and other concepts are defined as one sided limits. Suppose that M is our manifold with boundary ∂M and take $p \in \partial M \subseteq M$. Then we know $T_p(\partial M) \subseteq T_pM$ has dimension $n - 1$, so by dimensional arguments the subspace

$$N_p(\partial M) := \{v \in T_pM \mid \langle v, w \rangle_g = 0 \text{ for all } w \in T_p(\partial M)\}$$

has dimension one. We call $N_p(\partial M)$ the normal subspace to ∂M and it turns out that we can choose a smooth vector field, with vectors lying inside $N_p(\partial M)$, that respects the induced orientation on the boundary (as given on an orientable manifold M by Proposition 15.24 of [4])

Proposition 2.12. *There is a unique smooth outward-pointing unit normal vector field N along ∂M.*

Proof. This is Proposition 15.33 in [4]. □

Theorem 2.13 (Divergence Theorem)**.** *Let M have boundary ∂M, and let N be the unit normal vector field of the previous proposition. Then if $X \in \mathfrak{X}(M)$ we have*

$$\int_M (\text{div}X) \, \text{d}V_g = \int_M (\text{div}X) \, \omega_g = \int_{\partial M} \langle X, N \rangle_g \, \text{d}V_{\tilde{g}},$$

where $\tilde{g}$ is the metric restricted to ∂M.

Proof. We have from Stoke's theorem that

$$\int_M (\text{div}X) \, \text{d}V_g = \int_M \text{d}(\iota_X \omega_g) = \int_{\partial M} \iota^*_{\partial M}(\iota_X \omega_g).$$

Take $X^T := X - \langle X, N \rangle_g N$ so that $X = \langle X, N \rangle_g N + X^T$. But note that $X^T_p \in T_p(\partial M)$ since X^T_p is normal to N_p. Then because $T_p(\partial M)$ is $(n-1)$-dimensional and $\omega_g \in \Omega^n(M)$, this means $\iota^*_{\partial M}(\iota_{X^T} \omega_g) = 0$ by dimensionality. Thus by linearity

$$\iota^*_{\partial M}(\iota_X \omega_g) = \iota^*_{\partial M}(\iota_{\langle X,N \rangle_g N} \omega_g) = \langle X, N \rangle_g \iota^*_{\partial M}(\iota_N \omega_g).$$

We note that $\iota^*_{\partial M}(\iota_N \omega_g) \in \Omega^{n-1}(\partial M)$, so that $\iota^*_{\partial M}(\iota_N \omega_g) = a\omega_{\tilde{g}}$ for some $a \in \mathbb{R}$. Noting that if $E_1, \ldots, E_{n-1}$ is an oriented orthonormal frame for ∂M then $N, E_1, \ldots, E_{n-1}$ is an oriented orthonoromal frame for M we see $a = 1$. Then we see $\iota^*_{\partial M}(\iota_X \omega_g) = \langle X, N \rangle_g \, \text{d}V_{\tilde{g}}$ and we have the theorem. □

Proposition 2.14. *Let M, N be as in Theorem 2.13. If $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$, then*

$$\mathrm{div}(fX) = f\mathrm{div}X + \langle \mathrm{grad}\ f, X\rangle_g.$$

From this and the divergence theorem we have the integration by parts formula

$$\int_M \langle \mathrm{grad}\ f, X\rangle_g\ \mathrm{d}V_g = \int_{\partial M} f\langle X, N\rangle_g\ \mathrm{d}V_{\tilde{g}} - \int_M (f\mathrm{div}X)\ \mathrm{d}V_g.$$

Proof. This is straightforward from the definitions and the divergence theorem. □

Though the integration by parts formula will play a bigger role than the divergence theorem in our later proofs, it is worth noting the divergence theorem's physical significance. The divergence theorem allows us to relate the divergence of a vector field $X \in \mathfrak{X}(M)$ to the degree to which the field is exiting the the manifold, since $\langle X, N\rangle_g$ measures to what degree the field X is normal to the manifold at any given point. Since we can extend this to any embedded submanifold in M, this allows us to think of the divergence as a measure of how much the field is exiting any given region, just as Gauss's theorem allowed us to think of the divergence in $\mathbb{R}^3$ this way.

Aside from any physical interpretations of the gradient and divergence we note that since the gradient is a map from $C^\infty(M)$ to $\mathfrak{X}(M)$ and the divergence gives a map back from $\mathfrak{X}(M)$ to $C^\infty(M)$ we can compose them to give a map from $C^\infty(M)$ back to itself. Though it is not initially obvious, this map encodes a lot of geometric information about our Riemannian manifold.

Definition 2.15 (Laplacian)**.** Define $\Delta_g \colon C^\infty(M) \to C^\infty(M)$ by

$$\Delta_g u := -\mathrm{div}(\mathrm{grad}\ u).$$

We call Δ_g the Laplacian on our manifold.

It is immediate from the expressions in coordinates for the gradient and divergence that the Laplacian is given in local coordinates as

$$\Delta_g u = -\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^j}\left(g^{ij}\sqrt{\det g}\frac{\partial u}{\partial x^i}\right),$$

for $u \in C^\infty(M)$. Note that in $\mathbb{R}^n$ since $g_{ij} = \delta_{ij}$, then $g^{ij} = \delta_{ij}$, and so $\Delta_{\mathbb{R}^n} u = -\sum_{i=1}^n \partial^2_{x^i} u$. Therefore $\Delta_{\mathbb{R}^n}$ is the standard Laplacian as mentioned in the introduction.

Example 2.16. In $\mathbb{S}^2$ we have that

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \quad \text{thus} \quad (g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1/\sin^2\theta \end{pmatrix}.$$

Therefore,

$$\Delta_{\mathbb{S}^2} u = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2},$$

in coordinates on our sphere.

Using Proposition 2.14 it is straightforward to verify the following.

Theorem 2.17 (Green's Theorem)**.** *Let $u, v \in C^\infty(M)$ with M, N and $\tilde{g}$ as before, then we have*

$$\int_M u\Delta_g v\ \mathrm{d}V_g = \int_M \langle \mathrm{grad}\ u, \mathrm{grad}\ v\rangle_g\ \mathrm{d}V_g - \int_{\partial M} u(Nv)\ \mathrm{d}V_{\tilde{g}}$$

2.3. **Normal Coordinates and Gauss Lemma.** One final class of results necessary for our proof of Weyl's law will be an investigation of geodesics and normal coordinates on a Riemannian manifold. In the interest of brevity we won't give much exposition or motivating examples. We refer the reader to [4], Chapters 4-6, for a more thorough treatment.

First we start by defining a notion of the derivative of a vector field using the metric on our Riemannian manifold M.

Definition 2.18. A Levi-Civita connection on M is a map

$$\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)\colon (X, Y) \mapsto \nabla_X Y$$

with the following properties:

(a) $\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y$ for all $f, g \in C^\infty(M)$, $X_1, X_2, Y \in \mathfrak{X}(M)$,

(b) $\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2$ for all $a, b \in \mathbb{R}$, $X, Y_1, Y_2 \in \mathfrak{X}(M)$,

(c) $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$ for all $f \in C^\infty(M)$, $X, Y \in \mathfrak{X}(M)$,

(d) $X\langle Y, Z\rangle_g = \langle \nabla_X Y, Z\rangle_g + \langle Y, \nabla_X Z\rangle_g$ for all $X, Y, Z \in \mathfrak{X}(M)$,

(e) $\nabla_X Y - \nabla_Y X = [X, Y]$ for all $X, Y \in \mathfrak{X}(M)$, where $[\cdot, \cdot]$ is the Lie bracket on $\mathfrak{X}(M)$.

Theorem 2.19. *Let M be our Riemannian manifold; there exists a unique Levi-Civita connection on M.*

Proof. This is Theorem 5.4 in [5]. □

We can then use this connection to define a notion of the directional derivative of a vector field along a curve γ. Define $D_t\colon \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$ to be the covariant derivative along γ, that is such that $D_t V = \nabla_{\gamma'(t)} \tilde{V}$ where $\tilde{V}$ is the extension of an extendible $V \in \mathfrak{X}(\gamma)$. By Lemma 4.9 in [5] this, along with the requirement that D_t is linear and satisfies the product rule, determines D_t uniquely.

Definition 2.20. A curve γ is called a geodesic if its acceleration is zero, that is if $D_t\gamma' \equiv 0$. We denote the geodesic through $V \in TM$ by γ_V.

Theorem 2.21 (Existence and Uniqueness of Geodesics)**.** *For any $p \in M$, any $V \in T_pM$, and any $t_0 \in \mathbb{R}$, there exists an open interval $I \subseteq \mathbb{R}$ containing t_0 and a geodesic $\gamma\colon I \to M$ satisfying $\gamma(t_0) = p$ and $\gamma'(t_0) = V$. Any two such geodesics agree where their domains overlap.*

Proof. The full proof can be found as Theorem 4.10 in [5]. □

Geodesics will turn out to have some very special geometric properties, one very important property being that they allow us to define a map between TM and M.

Definition 2.22. Let $\mathcal{E} := \{V \in TM \mid \gamma_V$ is defined on an interval containing $[0, 1]\}$ and define a map $\exp\colon \mathcal{E} \to M$ by $\exp(V) = \gamma_V(1)$.

Proposition 2.23. *The exponential has the following properties:*

(a) *$\mathcal{E}$ is an open subset of TM containing the zero section, and each set $\mathcal{E}_p$ is star-shaped with respect to 0,*

(b) *for each $V \in TM$ the geodesic γ_V is given by $\gamma_V(t) = \exp(tV)$ for all t such that each side is defined,*

(c) *for any* $p \in M$ *there is a neighborhood* W *of the origin in* T_pM *and a neighborhood* U *of* p *in* M *such that* $\exp_p\colon W \to U$ *is a diffeomorphism.*

Proof. This is found as Proposition 5.7 and Lemma 5.10 in [5]. □

We note that any orthonormal basis $\{v_i\}$ for T_pM will give an isomorphism between $\mathbb{R}^n$ and T_pM, $v\colon \mathbb{R}^n \to T_pM\colon x \mapsto x^i v_i$. With appropriate neighborhoods U of p we may combine this isomorphism with $\exp_p$ to get a natural coordinate chart, which we call normal coordinates. Normal coordinates greatly simplify many calculations, but for our purposes we will find it useful to introduce *polar* normal coordinates, which are analogous to spherical coordinates in the case of $\mathbb{R}^n$. Note that the map ϕ from $(0,\infty) \times \mathbb{S}^{n-1}$ to $T_pM \setminus \{0\}$ given by $\phi(r,\xi) = r \cdot \iota(\xi)^i v_i$ is a diffeomorphism, where $\iota\colon \mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^n$ is the inclusion map for $\mathbb{S}^{n-1}$ with angular coordinates ξ. By Proposition 2.23(c) we then know that for any $p \in M$ there exists some $\delta > 0$ and some neighborhood U of p such that $\rho\colon (0,\delta) \times \mathbb{S}^{n-1} \to U$ where $\rho = \exp_p \circ \phi|_{(0,\delta)\times\mathbb{S}^{n-1}}$ is a diffeomorphism.

Definition 2.24. Where appropriately defined, the coordinate chart $(U, \rho^{-1}) = (U, (r, \xi^i))$ is called polar normal coordinates centered at $p \in M$.

Polar normal coordinates have some very special properties, especially with regards to the radial coordinate.

Lemma 2.25 (Gauss Lemma)**.** *For every point* $q \in U \setminus \{p\}$ *the vector* $\partial/\partial r$ *is the velocity vector of the unit speed geodesic from* q *to* p*, and thus* $|\partial/\partial r|_g = 1$*. Additionally for every* $p \in M$ *the vector field* $\partial/\partial r$ *is orthogonal to the* $2(n-1)$ *dimensional subspace* $T(\rho(\mathfrak{r}, \mathbb{S}^{n-1})) \subseteq TM$ *provided* $\mathfrak{r} < \delta$*.*

Further, in polar coordinates the metric is given by $g = \mathrm{d}r^2 + g_{ij}\mathrm{d}\xi^i\mathrm{d}\xi^j$*, where* g_{ij}/r^2 *asymptotically approaches the components of the standard metric of* $\mathbb{S}^{n-1}$ *as* $r \to 0$*, with* $1 \leq i, j \leq n-1$*.*

Proof. The first paragraph is found as Proposition 5.11(e) and Theorem 6.8 of [5].

We see from the first sentence that $\langle \partial/\partial r, \partial/\partial r\rangle_g = 1$. By the definition of ρ the geodesic spheres $\rho(\mathfrak{r}, \mathbb{S}^{n-1})$ have tangent spaces spanned by $\partial/\partial\xi^i$ for $i = 1, \dots, n-1$, so the orthogonality condition means that $\langle \partial/\partial r, \partial/\partial \xi^i\rangle_g = 0$ for $i = 1, \dots, n-1$. Thus in coordinates we know that $g = \mathrm{d}r^2 + g_{ij}\mathrm{d}\xi^i\mathrm{d}\xi^j$ wherever polar normal coordinates are defined. From Proposition 5.11(c) in [5] we know that in normal coordinates, $(U, (x^i))$, $g = \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$ at $p \in M$ $(r = 0)$, so g_p takes the form of the standard metric in canonical coordinates on $\mathbb{R}^n$ at p. Since the transition map between normal and polar normal coordinates is the same as that between canonical and spherical coordinates on $\mathbb{R}^n$, then by the smoothness of g and the fact that $\mathrm{d}r^2 + r^2 g_{\mathbb{S}^{n-1}}$ is the standard metric on $\mathbb{R}^n$ in spherical coordinates, we have our result. □

This result has two important corollaries.

Corollary 2.26. *In polar normal coordinates, if* $h \in C^\infty(M)$ *depends only on* r*, then*

$$\langle \text{grad } h, \text{grad } f\rangle_g = \frac{\partial h}{\partial r}\frac{\partial f}{\partial r}$$

for any $f \in C^\infty(M)$*.*

Additionally the Laplacian takes the form

$$\Delta_g = -\frac{1}{\sqrt{\det g}} \frac{\partial}{\partial r} \left(\sqrt{\det g} \frac{\partial}{\partial r} \right) + \Delta_{\rho(r,\mathbb{S}^{n-1})}$$

where $\Delta_{\rho(r,\mathbb{S}^{n-1})}$ *is the Laplacian induced on the geodesic sphere* $\rho(r, \mathbb{S}^{n-1})$*, and depends only on the angular coordinates* ξ*.*

Proof. This follows directly from the expression for g in these coordinates, as given in the Gauss lemma, and from the expressions for the gradient and the Laplacian in local coordinates. □

Corollary 2.27. *In polar normal coordinates centered at* $x \in M$ *define*

$$D(y) = \sqrt{\det g(r, \xi)}/r^{n-1} \text{ where } y = (r, \xi) \in M.$$

Then there is a neighborhood U *of* x *such that* $D \colon U \setminus \{x\} \to (0, \infty)$ *is smooth. Further* D *goes asymptotically to a smooth function of only* ξ *as* $r \to 0$*.*

Proof. D is clearly smooth in some open set U around x, from its definition in polar normal coordinates. Further in $U \setminus \{x\}$ we have $r > 0$ so $D(y) < \infty$, and we also have that (g_{ij}) is a positive-definite symmetric matrix, so it has positive determinant. Thus $D \colon U \setminus \{x\} \to (0, \infty)$ is smooth. Finally we note that as $r \to 0$ then

$$\sqrt{\det g}/r^{n-1} \to \sqrt{\det(r^2 g_{\mathbb{S}^{n-1}})}/r^{n-1} = \sqrt{r^{2(n-1)} \det g_{\mathbb{S}^{n-1}}}/r^{n-1} = \sqrt{\det g_{\mathbb{S}^{n-1}}}$$

by the limiting argument in the Gauss lemma. Thus as $r \to 0$, $D(y) \sim \sqrt{\det g_{\mathbb{S}^{n-1}}}$ which is a smooth function of only the angular coordinates ξ. □

Finally, it is possible to use the Gauss lemma to determine perhaps the most important feature about polar normal coordinates.

Theorem 2.28. *Every minimizing curve is a geodesic with unit speed parametrization. Further, if* y *is within any geodesic ball centered at* $x \in M$*, then in polar normal coordinates we have* $r = d_g(x, y)$*.*

Proof. These are Theorem 6.6 and Corollary 6.11 in [5]. □

Note the geometric importance of this information. We were able to use the existence of geodesics to construct an analog to spherical coordinates on our manifolds. Using these coordinates there is one that is singled out as giving the unique shortest distance between two points. This will be very helpful to us later.

3. Weyl's Law

We have now built up or reviewed enough background to address Weyl's law in its generalized form on our Riemannian manifold. However, in order to prove it without diving into the theory of pseduo-differential operators [7] we will later use an approximation technique that involves normal coordinates and which in physics is often called the DeWitt expansion [3]. In order to do this in a succinct way, we will now make the restriction to manifolds without boundary, though with sufficiently nice boundary conditions it turns out that Weyl's law still holds [7]. Note that $\mathbb{S}^2$ (and in fact $\mathbb{S}^n$) remains an example Riemannian manifold that is compact, connected, oriented, and has empty boundary.

Just as in the introduction, we define the asymptotic distribution

$$N(\lambda) := |\{\lambda_k \leq \lambda \mid \lambda_k \text{ an eigenvalue of } \Delta_g\}|,$$

for the case $\partial M = \varnothing$. We wish to prove the following result.

Theorem 3.1 (Weyl's Law). *If M is a compact connected oriented Riemannian manifold then*

$$N(\lambda) \sim \frac{|B_n|}{(2\pi)^n} \mathrm{Vol}(M) \lambda^{n/2},$$

where $|B_n|$ is the volume of the n-ball $B_n(0) \subseteq \mathbb{R}^n$.

In order to prove Weyl's Law, we shift focus to look at an extension of the heat equation on $\mathbb{R}^n$ to our Riemannian manifold M. Investigating the heat equation will provide deep insight into the behavior of the eigenvalues of Δ_g on our manifold. It is worth stopping in order to attempt to provide some physical intuition as to why this should be. Suppose that we have a compact region $D \subseteq \mathbb{R}^n$. Now set the boundary of D to some constant temperature field, and give the region an initial temperature. If the temperature on the region evolves according to the heat equation, then after waiting a sufficiently long time the temperature should satisfy Laplace's equation on D, with boundary conditions dictated by the constant temperature field. The approach of the temperature on D to a temperature field that satisfies Laplace's equation is exponential and is governed by the eigenvalues of $\Delta_{\mathbb{R}^n}$. Similarily, paying attention to the way solutions of the heat equation decay on our manifold M will tell us information about the eigenvalues of Δ_g on M. This connection between the decay of solutions to the heat equation and eigenvalues of Δ_g helps explain our interest in defining a heat equation on M.

First we define what the heat equation is, and then what a fundamental solution of this equation is. When proving that such a fundamental solution exists on our manifold, we will see that it encodes geometric information about the manifold (like its dimension and volume). However, just like solutions of the heat equation in $\mathbb{R}^n$, its decay will encode information about the eigenvalues of Δ_g. The fact that the fundamental solution encodes both geometric information about the manifold and information about the eigenvalues of Δ_g will allow us to connect the eigenvalues of Δ_g to geometric information about M, in the form of Weyl's law. Before we can see this beautiful connection we must first explore the extension of the heat equation on $\mathbb{R}^n$ to our manifold M.

Definition 3.2 (Homogenous Heat Equation). Let the operator $L = \Delta_g + \partial/\partial t$ act on functions $u\colon M \times (0, \infty)\colon (x, t) \mapsto u(x, t)$ which are C^2 in x and C^1 in t. We define the homogenous heat equation with initial data $f \in L^2(M)$ to be

$$\begin{cases} Lu(x,t) = 0 & (x,t) \in M \times (0,\infty) \\ u(x,0) = f(x) & x \in M \end{cases},$$

and say that a function u satisfies the homogenous heat equation with initial data f if it fulfills these conditions.

The physicist may be familiar with Green's functions as used to solve differential equations in generality. Here we make an identical definition, though we follow the analysts and call such an object the fundamental solution.

Definition 3.3. A fundamental solution of the heat equation is a function $p\colon M \times M \times (0,\infty) \to \mathbb{R}$ which is C^2 on M, C^1 on $(0,\infty)$, and such that

$$L_y p = 0 \quad \text{and} \quad \lim_{t\to 0} p(x,y,t) = \delta(y-x),$$

where δ is the Dirac delta distribution.

As is often the case we can gleam some valuable geometric intuition from the formulation of our problem on $\mathbb{R}^n$, despite the fact that $\mathbb{R}^n$ is not compact.

Proposition 3.4. *The function $p\colon \mathbb{R}^n \times \mathbb{R}^n \times (0,\infty) \to [0,\infty]$ given by*

$$p_{\mathbb{R}^n}(x,y,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\|x-y\|^2/4t}$$

is a fundamental solution to the heat equation, as we have defined it, on all of $\mathbb{R}^n$.

Proof. This is a well-known result. It is intutively clear that the exponential piece $e^{-\|x-y\|^2/4t}$ will produce something looking like the delta distribution as $t \to 0$. The factor of $(4\pi t)^{-n/2}$ is tacked on to ensure this does in fact go to the delta distribution as $t \to 0$. That $L_y p_{\mathbb{R}^n} = 0$ follows immediately from a computation. □

Before we try to prove existence of a fundamental solution in generality, we first want to understand whether or not it is unique. We cite the following result, which follows directly from a result known as Duhamel's principle.

Proposition 3.5. *The fundamental solution to the heat equation, if it exists, is unique and unchanged by a permutation of its space variables.*

Proof. This is Proposition 36 of Chapter 6 in [2] □

Having established uniqueness we now want to use analogy with the fundamental solution on $\mathbb{R}^n$ to construct a fundamental solution on our manifold M. We might try to construct a similar function on our manifold M, replacing the standard Euclidean metric $\|x-y\|$ with our new metric $d_g(x,y)$. However, in order to make sense of differentiation in this case, we want to be in a coordinate system with a coordinate which is equal to the geodesic distance $d_g(x,y)$. Thankfully this is exactly what our discussion of polar normal coordinates has prepared us to do. We then try an Ansatz solution which resembles $p_{\mathbb{R}^n}$, but with use of polar normal coordinates. However, along the way we will have to be slightly more clever, since our use of normal coordinates means that we will have to be careful with domains of definition.

Theorem 3.6. *There exists $\delta > 0$ such that the fundamental solution p exists, with $p \in C^\infty(M \times M \times (0,\infty))$, and has expansion*

$$p(x,y,t) = \frac{1}{(4\pi t)^{n/2}} e^{-d_g(x,y)^2/4t} \left(\sum_{j=0}^{k} t^k u_k(x,y) + O(t^{k+1}) \right),$$

for every $x, y \in M$ such that $d_g(x,y) < \delta$. Additionally $u_0(x,x) = 1$ for all $x \in M$.

Proof. Our proof will follow several steps. The first will be to produce the asymptotic expansion of $p(x,y,t)$ where it is purported to hold. The next steps will be to extend these local asymptotics to the whole manifold.

Take $\delta > 0$ to be such that $\exp_x$ is a diffeomorphism from $B_{2\delta}(0) \subset T_xM$ onto its image. Note that because our manifold M is compact we can choose this δ to be such that $\exp_x$ is a diffeomorphism from $B_{2\delta}(0)$ for all $x \in M$. Now let $V_{2\delta} = \{(x,y) \mid d_g(x,y) < 2\delta\} \subset M \times M$ and define the function

$$G\colon V_{2\delta} \times (0,\infty) \to [0,\infty) \text{ to be such that } G(x,y,t) = \frac{1}{(4\pi t)^{n/2}} e^{-d_g(x,y)^2/4t},$$

so that G has precisely the form we would expect of a fundamental solution, by analogy with $\mathbb{R}^n$. Now we construct

$$S_k\colon V_{2\delta} \times (0,\infty) \to \mathbb{R}\colon (x,y,t) \mapsto G(x,y,t)(u_0(x,y) + tu_1(x,y) + \cdots + t^k u_k(x,y)),$$

where we wish to find $u_j \in C^\infty(V_{2\delta})$ for $0 \le j \le k$ such that $L_y S_k = t^k G \Delta_g u_k$. If we then take k large enough, this will allow us to take $L_y S_k$ to zero as $t \to 0$, getting us a step closer to a solution which satisfies the homogenous heat equation for all t.

It is easy to check from the definition of the Laplacian that if $u, v \in C^\infty(M)$, then

$$\Delta_g(uv) = u\Delta_g v - 2\langle \text{grad } u, \text{grad } v\rangle_g + v\Delta_g u.$$

Using this product rule, the fact that G depends only on r, and the results of Corollary 2.26 for computing in polar normal coordinates, after a long series of computations we conclude that

$$L_y S_k = \left(\frac{\partial q}{\partial t} + \frac{r}{2t}\frac{q}{D}\frac{\partial D}{\partial r} + \Delta_g q + \frac{r}{t}\frac{\partial q}{\partial r} \right) G,$$

where we have taken q to be the sum $u_0 + tu_1 + \cdots + t^k u_k$ for brevity, and where D is as in Corollary 2.27. Separating this into powers of t we have

$$L_y S_k = \left[\frac{1}{t}\left(r\frac{\partial u_0}{\partial r} + \frac{r}{2D}\frac{\partial D}{\partial r} u_0 \right) + \sum_{j=1}^{k} t^{j-1} \left(r\frac{\partial u_j}{\partial r} + \left(\frac{r}{2D}\frac{\partial D}{\partial r} + j \right) u_j + \Delta_g u_{j-1} \right) + t^k \Delta_g u_k \right] G.$$

Thus we would like to choose the $u_j \in C^\infty(V_{2\delta})$ such that u_0 satisfies $r\partial_r u_0 + r\partial_r D u_0/(2D) = 0$ and

$$r\frac{\partial u_j}{\partial r} + \left(\frac{r}{2D}\frac{\partial D}{\partial r} + j \right) u_j + \Delta_g u_{j-1} = 0 \quad \forall j = 1, \ldots, k. \tag{1}$$

We may choose $u_0(x,y) = f(\xi)D^{-1/2}(y)$, as a quick look will reveal that this satisfies the equation on u_0, and is smooth by Corollary 2.27. By Corollary 2.27 we have that $D(y)/h(\xi) \to 1$ as $y \to x$, where h is some smooth function of the angular coordinates ξ. Then we choose $f = h^{1/2}$ to be a smooth function, and note that as $y \to x$ we then have that $u_0(x,y) \to 1$. Thus we see $u_0(x,y) = h^{1/2}(\xi)D^{-1/2}(y)$ satisfies the desired equation and is smooth for all $x, y \in M$, provided we take $u_0(x,x) = 1$ for all $x \in M$.

If we now take $u_j = fr^{-j}D^{-1/2}$, with f a function of r only, we see that it will satisfy (1), provided $\partial f/\partial r = \mathrm{d}f/\mathrm{d}r = -D^{1/2}r^{j-1}\Delta_g u_{j-1}$. Since this is an ODE we may integrate with respect to r to find f. Noting from Theorem 2.28 that r is the coordinate along the geodesic γ from x to y we take:

$$u_j(x,y) = -\frac{1}{(d_g(x,y))^j D^{1/2}(y)} \int_0^{d_g(x,y)} D^{1/2}(\gamma(s)) s^{j-1} \Delta_g u_{j-1}(\gamma(s), y)\ \mathrm{d}s.$$

Since u_0 is smooth we see by induction that $u_j \in C^\infty(V_{2\delta})$ for $0 \le j \le k$. Note that this means S_k will be smooth on $V_{2\delta} \times (0,\infty)$ since G is smooth here as well. We have therefore achieved our goal and $L_y S_k = t^k G \Delta_g u_k$ for these choices of u_j. Lastly we comment on the physical significance here. Since D depends only on geometric information, that is only on the determinant of the metric tensor and on geodesics, the u_j are entirely determined by the geometric information of M. In fact a more detailed look would reveal that $u_1(x,x) = \kappa(x)/6$, where κ is the scalar curvature [2].

Having appropriately constructed the asymptotics of our candidate fundamental solution in a small neighborhood around every $x \in M$, we now want to extend this to the whole manifold. We start naively extending using a bump function, in the hopes that we can then further modify the resulting function into a fundamental solution. Take α to be a smooth bump function on $M \times M$ with $\alpha \equiv 1$ on V_δ and $\alpha \equiv 0$ on $V_{2\delta}^c$. Let $H_k = \alpha S_k$, so that $H_k \in C^\infty(M \times M \times (0,\infty))$, since S_k is defined where $\alpha \not\equiv 0$. Then we have two important results, which we sketch the proof of.

Lemma 3.7. *For $k > n/2$ we have that $L_y H_k \in C^l(M \times M \times [0,\infty))$ for every l such that $0 \le l < k - n/2$. Additionally for all $x, y \in M$ we have that $\lim_{t\to 0} H_k(x,y,t) = \delta(x-y)$.*

Proof. Note that to prove the first condition the only thing necessary is to show that it holds when H_k is extended to $t = 0$. Suppose $(x,y) \in V_\delta$, so that $\alpha \equiv 1$, then

$$L_y H_k = L_y S_k = \frac{t^{k-n/2}}{(4\pi)^{n/2}} e^{-d_g(x,y)^2/4t} \Delta_g u_k.$$

We note that this goes to zero for any $(x,y) \in V_\delta$ as $t \to 0$, since $k > n/2$, so we may extend $L_y H_k$ on $V_\delta \times [0,\infty)$ by taking it to be zero at $t = 0$. Similarily since $H_k = 0$ on $V_{2\delta}^c \times (0,\infty)$ we may extend $L_y H_k$ so that it is zero at $t = 0$. Finally on $V_\delta^c \cap V_{2\delta} \times (0,\infty)$ we will have that $r = d_g(x,y) \ge \delta$, so that $L_y H_k$ goes to zero as $t \to 0$ regardless of the order of the pole at $t = 0$, since $L_y H_k$ will have a factor of $e^{-d_g(x,y)^2/4t}$ multiplying each term. Thus we can extend $L_y H_k$ to be zero at $t = 0$ in this case as well. Looking over the various powers of t we see $L_y H_k \in C^l(M \times M \times [0,\infty))$ for $0 \le l < k - n/2$.

Let $f \in C^0(M)$, we see that:

$$\begin{aligned}\lim_{t\to 0} \int_M H_k(x,y,t) f(y)\ \mathrm{d}V_g(y) &= \lim_{t\to 0} \int_{B_\delta(x)} H_k(x,y,t) f(y)\ \mathrm{d}V_g(y) \text{ since } \lim_{t\to 0} G = 0, \text{ if } y \notin B_\delta(x) \\ &= \lim_{t\to 0} \sum_{j=0}^{k} t^j \int_{T_xM} \frac{1}{(4\pi t)^{n/2}} e^{-\|v\|^2/4t} u_j(x, \exp_x(v)) f(\exp_x(v))\ J(v)\mathrm{d}v\end{aligned}$$

where this last step was a change to normal coordinates with $J(v)$ as the Jacobian of this transformation and where we took u_j to be zero outside of $B_\delta(0) \subset T_xM$. Note that we could pass limits here by taking k large enough so that the continuity demonstrated above held. Then because $(4\pi t)^{-n/2} e^{-\|v\|^2/4t} = p_{\mathbb{R}^n}(0,v,t)$, as above, we see that as $t \to 0$ this goes to $f(x)$, since $u_0(x,x) = 1$. Thus we see that $\lim_{t\to 0} H_k(x,y,t) = \delta(x-y)$. □

This is somewhat remarkable. Simply by choosing a candidate solution geometrically similar to that found on $\mathbb{R}^n$ we have arrived at a solution which is close to satisfying the homogenous heat equation and has the correct behavior as $t \to 0$. We now want to find a way to modify this extension in such a way that it will satisfy the heat equation in its y-variable exactly, not just be C^l. In addition we want to maintain the asymptotics we constructed for S_k. We attempt to do this in a standard way seen in PDEs [7].

The convolution for $F, H \in C^0(M \times M \times [0,\infty))$ will be defined as

$$(F * H)(x,y,t) = \int_0^t \int_M F(x,z,s)H(z,y,t-s)\ \mathrm{d}V_g(z)\mathrm{d}s,$$

so as to convolve both time and space information. It is clear from any standard proof of convolutions preserving regularity that we will have $F * H_k \in C^l(M \times M \times [0,\infty))$ for all $F \in C^0(M \times M \times [0,\infty))$. This allows us to differentiate the convolution. Because of the integral with t some computation reveals

$$L_y(F * H_k)(x,y,t) = F(x,y,t) + (F * L_y H_k)(x,y,t).$$

Knowing that convolutions can often help control the size of results, we then might try a fundamental solution $p = H_k - F_k * H_k$. If we select the right F_k, then we might be able to preserve the time asymptotics of H_k with the first term, while the second term is kept small. At the same time we might be able to remove the remainder when we operate with L_y using the term $F_k * H_k$. Indeed suppose that F_k is twice differentiable, then we see that $L_y p = L_y H_k - L_y(F_k * H_k) = L_y H_k - F_k - F_k * L_y H_k$. This suggests that we take

$$F_k = \sum_{j=1}^{\infty} (-1)^{j+1} (L_y H_k)^{*j},$$

where ^{*j} indicates taking the convolution with itself j times, provided that this sum is defined. We now prove that this sum is in fact defined.

Lemma 3.8. *For $k > n/2$ and $0 \le l < k - n/2$ the series F_k is defined and $F_k \in C^l(M \times M \times [0,\infty))$. In addition for $t_0 > 0$ there exists a constant $C = C(t_0)$ such that*

$$\|F_k(\cdot,\cdot,t)\|_{L^\infty(M\times M)} \le C t^{k-n/2} \quad \textit{for all } t \in [0,t_0].$$

Proof. This is Lemma 49 in Chapter 6 of Canzani, and is left as an exercise since the lemma is short on content but long to write out. It follows from a similar bound of $\|L_y H_k\|_{L^\infty(M\times M\times[0,t_0])}$ and induction. □

From the lemma above it immediately follows that $p = H_k - F_k * H_k$ is a solution for the homogenous heat equation for all $k > n/2 + 2$, as we now demonstrate. We see this since the previous lemma demonstrates that $p = H_k - F_k * H_k \in C^2(M \times M \times [0,\infty))$ for these k, so that $L_y p$ is defined and $L_y p = L_y H_k - F_k - F_k * L_y H_k$. From the series that defines F_k we see $L_y p = 0$. Additionally we see that if $f \in C^0(M)$ and $k > n/2 + 2$, then since $F_k, H_k, p \in C^2(M \times M \times [0,\infty))$ and $\lim_{t\to 0} H_k(x,y,t) = \delta(x-y)$ we have:

$$\lim_{t\to 0} \int_M p(x,y,t)f(y)\ \mathrm{d}V_g(y) = f(x) - \lim_{t\to 0} \int_M (F_k * H_k)(x,y,t)f(y)\ \mathrm{d}V_g(y) = f(x),$$

since the last term is controlled by the L^∞ norm of F_k. Thus we see that $p = H_k - F_k * H_k$ is in fact a fundamental solution to the heat equation, provided $k > n/2 + 2$.

Now we know by Proposition 3.5 the fundamental solution is unique. Thus p is independent of k, and $H_k - F_k * H_k$ gives the same fundamental solution for any choice of k. Since $H_k - F_k * H_k \in C^{\lfloor k-n/2 \rfloor}(M \times M \times (0,\infty))$ for every $k > n/2$, then $p \in C^\infty(M \times M \times (0,\infty))$. From the error estimates on F_k found in the lemma above, and the fact that $H_k \in C^\infty(M \times M \times (0,t)) \subseteq L^1(M \times M \times (0,t))$ we may find a constant B such that $\|F_k * H_k(\cdot,\cdot,t)\|_{L^\infty(M\times M)} \le B t^{k+1-n/2}$ for all $t \le t_0$, for some t_0. This bound means that on

V_δ, that is where $d_g(x, y) < \delta$ and where $H_k \equiv S_k$, the asymptotics will be governed by the asymptotics of S_k, which we constructed to have the asymptotics asked for in the statement of the theorem. Thus we have constructed the fundamental solution on our manifold M and it has the desired asymptotics. $\square$

Now that we have constructed a fundamental solution, we are prepared to examine the consequences of such a solution.

Proposition 3.9. *For $f \in L^2(M)$ the function*

$$u(x,t) = \int_M p(x,y,t) f(y) \; \mathrm{d}V_g(y)$$

is smooth on $M \times (0, \infty)$ and is a solution to the homogenous heat equation with initial data f. Further, it is the unique such solution.

Proof. First we note that if we have $f \in L^2(M)$ then $f \in L^1(M)$ since M is compact. Thus f is locally L^1. Further p is C^∞ with compact support since M is compact. Taking $u(x,t) = \int_M p(x,y,t) f(y) \; \mathrm{d}V_g(y)$, we then see that $u \in C^\infty(M \times (0, \infty))$ by the extension to M of a well known theorem for convolutions (that is if $f \in L^1_{\text{loc}}(\mathbb{R}^k)$ and $\psi \in C_c^\infty(\mathbb{R}^k)$ then $f * \psi$ is C^∞). Note these conditions also justify differentiating under the integral sign, so that a simple computation reveals u satisfies the homogenous heat equation with initial condition given by f, by the properties of the fundamental solution. We further have, by differentiating under the integral sign, and Green's theorem with $\partial M = \varnothing$, that $\mathrm{d}/\mathrm{d}t \|u(\cdot,t)\|^2_{L^2(M)} = -2\|\text{grad } u(\cdot,t)\|^2_{L^2(M)} \leq 0$, with equality only when u is a constant. This decreasing of the L^2 norm allows us to conclude that a solution to the heat equation which is initially zero everywhere stays zero everywhere, implying uniqueness of solutions, since L is a linear operator. $\square$

Thus every solution to the homogenous heat equation can be expressed in this form, completing the analogy of a fundamental solution with a Green's function.

Definition 3.10. For $t > 0$ we define the heat propagator $e^{-t\Delta_g} \colon L^2(M) \to L^2(M)$

$$e^{-t\Delta_g} f = \int_M p(x,y,t) f(y) \; \mathrm{d}V_g.$$

Note that by Proposition 3.9, the heat propagator maps initial data to the unique solution to the homogenous heat equation.

The heat propagator has several important properties as an operator.

Lemma 3.11. *For $t, s > 0$ we have that $e^{-t\Delta_g} \circ e^{-s\Delta_g} = e^{-(t+s)\Delta_g}$. We further have that $e^{-t\Delta_g}$ is a self-adjoint, compact, and positive operator. Finally, for all $t \in \mathbb{R}_{>0}$ we have that $(e^{-\Delta_g})^t = e^{-t\Delta_g}$.*

Proof. This can be found as Lemma 41 in Chapter 6 of Canzani. Every argument is fairly straightforward, using as the most complicated tool the dominated convergence theorem, except for the proof that $e^{-t\Delta_g}$ is a compact operator. Proving this result is done by considering that the map $e^{-t\Delta_g} \colon L^2(M) \to H^1(M)$ is continuous, and the inclusion $H^1(M) \subset L^2(M)$ is compact, where $H^1(M)$ is an appropriately defined Sobolev space on our manifold M. This result is covered in most texts on PDE, and can be extended in a natural way to our manifold M. $\square$

This lemma leads us to an important decomposition of our fundamental solution.

Theorem 3.12. *On our manifold M there exists a complete orthonormal basis $\{\phi_0, \phi_1 \ldots\}$ of eigenfunctions of Δ_g, where we have that $\lambda_0 \leq \lambda_1 \leq \cdots \to \infty$. For every $j \in \mathbb{N}$ we have that $\phi_j \in C^\infty(M)$. Finally, the fundamental solution can be decomposed as*

$$p(x, y, t) = \sum_{j=0}^{\infty} e^{-\lambda_j t}\phi_j(x)\phi_j(y).$$

Proof. The operator $e^{-\Delta_g}$ is positive on the Hilbert space $L^2(M)$ and is also a compact self-adjoint operator on $L^2(M)$ by Lemma 3.11. Thus we know by the spectral theorem that there exists a complete orthonormal basis of $L^2(M)$ of eigenfunctions of $e^{-\Delta_g}$, $\{\phi_0, \phi_1, \ldots\}$, whose eigenvalues $\beta_0 \geq \beta_1 \geq \cdots \to 0$. Since we have that $e^{-t\Delta_g} = (e^{-\Delta_g})^t$ we see that $e^{-t\Delta_g}\phi_j = \beta_j^t\phi_j$ for all $j \in \mathbb{N}_0$. Note that by Proposition 3.9 $e^{-t\Delta_g} \colon L^2(M) \to C^\infty(M) \subseteq L^2(M)$ for any fixed $t > 0$, because M is compact and p is smooth. Thus since $\beta_j^t\phi_j = e^{-t\Delta_g}\phi_j \in C^\infty(M)$ for $t > 0$, we conclude $\phi_j \in C^\infty(M)$ for all $j \in \mathbb{N}_0$. Then we know that $\phi_0 \in C^\infty(M) \subseteq L^2(M)$, so we have that

$$u(x, t) = \int_M p(x, y, t)\phi_0(y)\ \mathrm{d}V_g(y) = e^{-t\Delta_g}\phi_0(x) = \beta_0^t\phi_0(x)$$

satisfies the homogenous heat equation with initial data ϕ_0, by Proposition 3.9. But then $\|u(\cdot, t)\|^2_{L^2(M)} = \beta_0^{2t}$ since the ϕ_j form an orthonormal basis. From the same proposition we know that this norm decreases with time, so we conclude $1 \geq \beta_0 \geq \beta_1 \geq \cdots \to 0$.

We then define the sequence $\lambda_k = -\ln\beta_k$ where we will therefore have $0 \leq \lambda_0 \leq \lambda_1 \leq \cdots \to \infty$. We note the relation derived above then becomes $e^{-t\Delta_g}\phi_j(x) = e^{-\lambda_j t}\phi_j(x)$. But since $\phi_j \in L^2(M)$ then $e^{-\Delta_g}\phi_j(x)$ solves the homogenous heat equation with initial data ϕ_j. This means that

$$0 = L\left(e^{-t\Delta_g}\phi_j(x)\right) = L\left(e^{-\lambda_j t}\phi_j(x)\right) = (\Delta_g\phi_j - \lambda_j\phi_j)e^{-\lambda_j t} \quad \forall t > 0,$$

so we may deduce that ϕ_j is an eigenfunction of Δ_g with eigenvalue λ_j. Thus the $\{\phi_0, \phi_1, \ldots\}$ are a complete orthonormal basis of $L^2(M)$ of eigenfunctions of Δ_g, as well as $e^{-\Delta_g}$. The expansion for p follows directly from the fact that $L^2(M)$ is a Hilbert space with basis $\{\phi_0, \phi_1, \ldots\}$, and $\int_M p(x, y, t)\phi_j(y)\ \mathrm{d}V_g(y) = e^{-\lambda_j t}\phi_j(x)$. □

Corollary 3.13. *For $t > 0$*

$$\int_M p(x, x, t)\ \mathrm{d}V_g(x) = \sum_{j=0}^{\infty} e^{-\lambda_j t}$$

Proof. This is immediate from the expansion of p above, and the fact that $\|\phi_j\|_{L^2(M)} = 1$ for every $j \in \mathbb{N}_0$. □

The expression above is very powerful, and it is worth stopping to mention what has resulted from our concentrated efforts to probe the fundamental solution. First we were able to find an expression for the fundamental solution which had asymptotics with coefficients that encoded geometric information about the manifold (Theorem 3.6), and which was controlled by $G(x, y, t)$, an exponential function that we came to by physical intuition and analogy with $\mathbb{R}^n$ (Proposition 3.4). Now after some further argument about this fundamental solution, we have realized that we can expand it in terms of smooth eigenfunctions of Δ_g and their eigenvalues, and that by integrating it we can get deep information about these eigenvalues. The hope then is that since we have one expression for p which encodes geometric information, at least asymptotically, and one expression which encodes the eigenvalues of Δ_g, then we can link these two expressions to determine how the eigenvalues encode geometric information about M. Indeed, doing just this will give us Weyl's law, though we need one final lemma.

Lemma 3.14 (Karamata)**.** *Suppose that μ is a positive measure on $[0, \infty)$ and $\alpha > 0$. Then:*

$$\lim_{t\to 0} t^\alpha \int_0^\infty e^{-tx}\, \mathrm{d}\mu(x) = a \implies \lim_{\lambda\to\infty} \lambda^{-\alpha} \int_0^\lambda \mathrm{d}\mu(x) = \frac{a}{\Gamma(\alpha+1)}$$

Proof. This is Theorem 71 of Chapter 7 of Canzani. Note that the physicist will be familiar with this as a Laplace transformation. □

We now restate and prove Theorem 3.1, after having developed the machinery to do so.

Theorem 3.15. *If M is a compact connected oriented Riemannian manifold then*

$$N(\lambda) \sim \frac{|B_n|}{(2\pi)^n} \mathrm{Vol}(M) \lambda^{n/2},$$

where $|B_n|$ is the volume of the n-ball $B_n(0) \subseteq \mathbb{R}^n$.

Proof. First we see that for $t > 0$:

$$\begin{aligned}
\sum_{j=0}^\infty e^{-\lambda_j t} &= \int_M p(x,x,t)\, \mathrm{d}V_g(x) \text{ by Corollary 3.13} \\
&= \int_M \frac{1}{(4\pi t)^{n/2}} e^{-d_g(x,x)^2/4t} \left(\sum_{j=0}^k t^k u_k(x,x) + O(t^{k+1}) \right) \mathrm{d}V_g(x) \text{ by Theorem 3.6} \\
&= \frac{1}{(4\pi t)^{n/2}} \left(\int_M u_0(x,x)\, \mathrm{d}V_g(x) + O(t) \right) \\
&= \frac{1}{(4\pi t)^{n/2}} \left(\mathrm{Vol}(M) + O(t) \right) \text{ by Theorem 3.6.}
\end{aligned}$$

Let $\mu = \sum \delta_{\lambda_j}$, that is let μ be the counting measure for the eigenvalues of Δ_g. Then using the eigenvalues' distribution as discussed in Theorem 3.12, we will have that:

$$\lim_{t\to 0} t^{n/2} \sum_{j=0}^\infty e^{-\lambda_j t} = \lim_{t\to 0} t^{n/2} \int_0^\infty e^{-tx}\, \mathrm{d}\mu(x)$$

By the asymptotics above we see that

$$\begin{aligned}
\lim_{t\to 0} t^{n/2} \int_0^\infty e^{-tx}\, \mathrm{d}\mu(x) &= \mathrm{Vol}(M)/(4\pi)^{n/2} \\
\implies \frac{\mathrm{Vol}(M)}{(4\pi)^{n/2}\Gamma(n/2+1)} &= \lim_{\lambda\to\infty} \lambda^{-n/2} \int_0^\lambda \mathrm{d}\mu(x) \text{ by Lemma 3.14} \\
&= \lim_{\lambda\to\infty} \lambda^{-n/2} N(\lambda),
\end{aligned}$$

since μ is the counting measure. But noting that $|B_n| = \pi^{n/2}/\Gamma(n/2+1)$ it is immediate that

$$N(\lambda) \sim \frac{|B_n|}{(2\pi)^n}\text{Vol}(M)\lambda^{n/2},$$

in the limit of large λ. □

Just as we had hoped we were able to use the fact that p encoded both geometric information about our Riemannian manifold M as well as information about the eigenvalues of Δ_g on M to deduce a connection between the geometry of M and the eigenvalues of Δ_g. In fact we see that the distribution of eigenvalues of Δ_g tells us both the dimension and the volume of our Riemannian manifold. Indeed, if we cared to investigate further we could deduce the total curvature, among other geometric information, from the spectrum of the Laplacian [2]. We have thus reached the end of our introduction to Riemannian geometry and our account of this striking proof laying at the center of one of the biggest revolutions in physics.

4. Acknowledgements

I would like to thank my mentors, Sean Howe and Yun Cheng, for their patience in working with me as I learned the material. I would like to thank Sean especially: for his thought provoking discussions about deep results in Riemannian geometry, for suggestions of additional material to study, and for suggesting this project in the first place! I would also like to thank Professor Irvine of the JFI for allowing me to take time off from my superfluid simulations in order to study Riemannian geometry. Finally, I would of course like to thank Professor May for setting up the REU and giving me a structure within which I can study math that really interests me.

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